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M-theory on eight-manifolds revisited: $\mathcal{N} = 1$ supersymmetry and generalized Spin(7) structures

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Abstract: The requirement of $\mathcal{N} = 1$ supersymmetry for M-theory backgrounds of the form of a warped product $\mathcal{M} \times_w X$, where X is an eight-manifold and $\mathcal{M}$ is three-dimensional Minkowski or AdS space, implies the existence of a nowhere-vanishing Majorana spinor ξ on X. ξ lifts to a nowhere-vanishing spinor on the auxiliary nine-manifold $Y := X \times S^1$, where S^1 is a circle of constant radius, implying the reduction of the structure group of Y to Spin(7). In general, however, there is no reduction of the structure group of X itself. This situation can be described in the language of generalized Spin(7) structures, defined in terms of certain spinors of $\mathrm{Spin}(TY \oplus T^*Y)$. We express the condition for $\mathcal{N} = 1$ supersymmetry in terms of differential equations for these spinors. In an equivalent formulation, working locally in the vicinity of any point in X in terms of a 'preferred' Spin(7) structure, we show that the requirement of $\mathcal{N} = 1$ supersymmetry amounts to solving for the intrinsic torsion and all irreducible flux components, except for the one lying in the **27** of Spin(7), in terms of the warp factor and a one-form L on X (not necessarily nowhere-vanishing) constructed as a ξ bilinear; in addition, L is constrained to satisfy a pair of differential equations. The formalism based on the group Spin(7) is the most suitable language in which to describe supersymmetric compactifications on eight-manifolds of Spin(7) structure, and/or small-flux perturbations around supersymmetric compactifications on manifolds of Spin(7) holonomy.

Keywords: M-Theory, Flux compactifications, Superstring Vacua.

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Contents

1. Introduction

It has been observed (starting with [1]), in connection to supergravity compactifications, that the concept of G-structures is a natural generalization of special-holonomy to the case where fluxes are present. Supersymmetry implies the existence of a nowhere-vanishing spinor on the internal manifold X, thereby reducing the structure group of X to G. In the presence of fluxes, X is no longer special-holonomy and the spinor is no longer covariantly constant: its failure to be such is parametrized by the (flux-dependent) intrinsic torsion of the Levi-Civita connection associated with the G-invariant metric on X. Moreover, the intrinsic torsion can be decomposed in irreducible G-modules, giving a characterization of X.

More recently, it was realized [2 – 7] that generic spinor Ansätze for the supersymmetry parameter naturally lead to the concept of generalized G-structures [8]. Roughly-speaking, generalized G-structures arise as follows: typically there will be two nowhere-vanishing spinors $\epsilon^\pm$ in the Ansatz for the supersymmetry parameter, each one inducing a reduction of the structure group to a subgroup $G_\pm$. Noting, in addition, that there is an isomorphism between bispinors on X and spinors of $TX \oplus T^*X$, we conclude that $\rho := \epsilon^+ \otimes \epsilon^-$ (which can also be thought of — by Fierzing — as a sum of forms on X) induces a reduction of the structure group of $TX \oplus T^*X$ to $G_+ \times G_- \subset \mathrm{Spin}(TX \oplus T^*X)$.

In addition to the reduction of the structure group of $TX \oplus T^*X$, supersymmetry implies that ρ should satisfy certain differential equations. For type II supergravities and for X a six- or seven-dimensional manifold, these equations have been identified, in the case where the Ramond-Ramond fields are zero, with certain integrability conditions for the generalized structures [2, 4]. Recently it has been possible to give a satisfactory mathematical description of the RR forms [7] by a generalization of the Hitchin functional [9] in which the RR forms appear as constraints. The role of the Hitchin functional in various topological theories is explored in [10 – 14].

Although a great deal is known about the connection of supersymmetry to generalized structures in six and seven dimensions, the case of eight-dimensional manifolds remains rather obscure (see however [15, 16]). In the present paper we wish to remedy the situation by examining the conditions for the most general $\mathcal{N} = 1$ three-dimensional AdS or Minkowski vacua in M-theory. An immediate consequence of supersymmetry is that associated with the eight-dimensional internal manifold X there is a nine-manifold $Y := X \times S^1$, such that Y supports a generalized Spin(7) structure on the sum of its tangent and cotangent bundles. The structure is given in terms of certain bispinors (spinors of $\mathrm{Spin}(9, 9)$) which are constrained to satisfy certain differential equations.

In a more conventional (equivalent) formulation, we show that $\mathcal{N} = 1$ supersymmetry implies the existence of a nowhere-vanishing *Majorana* spinor on X. This lifts to a nowhere-vanishing spinor on $Y = X \times S^1$ and hence implies the reduction of the structure group of Y to Spin(7). Note that, in general, X does *not* support nowhere-vanishing *Majorana-Weyl* spinors and the structure group of X itself is *not*, in general, reduced. However, working locally in an open set of X, one can still decompose all fields in terms of irreducible Spin(7)-modules. We are then able to solve for the intrinsic torsion and all irreducible flux components, except for the one lying in the **27** of Spin(7) which cannot be determined by the supersymmetry equations, in terms of the warp factor and a certain one-form L on X (constructed as a ξ bilinear). L is not necessarily nowhere-vanishing, unless the structure group is further reduced to G_2. In addition, L is constrained to satisfy a pair of differential equations.

The case examined here is an alternative formulation to the works of [17, 18], which assume the existence of at least one nowhere-vanishing Majorana-Weyl spinor, as well as of [19] (see also [20]). The authors of reference [19] perform their analysis by decomposing the flux and the intrinsic torsion of the internal manifold in terms of irreducible G_2 representations. These decompositions are valid only outside the zero locus of both Majorana-Weyl spinors $\xi^\pm$, where $\xi = \xi^+ \oplus \xi^-$, and they break down at the points where either of $\xi^{\pm}$ vanishes. The analysis of [19] is valid globally only if there is a further reduction of the structure group of X to G_2. Note, however, that in open sets where neither of $\xi^{\pm}$ vanishes, the analyis of [19] is perfectly sufficient to determine the most general local form of the geometry. In such opens sets, our formulæ in section 3.1 below should reduce to the corresponding formulæ given in [19], to the extent they overlap (some of the flux components were not given explicitly in [19]).

There are certain cases in which it is advantageous to work with a Spin(7) rather than with a G_2 structure. Clearly this is true if one wishes to consider supersymmetric M-theory compactifications on eight-manifolds with a global Spin(7) structure. A (very) special case thereof is compactifications on eight-manifolds with Spin(7) holonomy. Also, even in the generic case of compactifications on eight-manifolds X where there is no reduction of the structure group of X to Spin(7), it is still advantageous to work with a Spin(7) strucutre if one wishes to describe the local geometry in the vicinity of a point where either of $\xi^{\pm}$ vanishes. Put in another way: let P be an arbitrary point in X. There is always an open set $U_P \subset X$ such that $P \in U_P$ and at least one of $\xi^{\pm}$ is nowhere-vanishing in U_P. I.e. for any point P, there is always an open patch U_P containing P, such that the decomposition in Spin(7) modules is valid in U_P. The same is *not* true for G_2, as can be seen by taking P to be a point where one of $\xi^{\pm}$ vanishes.

The plan of this paper is as follows: section 2 includes some general background on G-structures and, more particularly, Spin(7)-structures. In section 3 we perform the supersymmetry analysis in terms of (ordinary) Spin(7)-structures on X. As an explicit example we have also included a small-flux perturbation around the special-holonomy solution involving the non-compact Spin(7)-manifold of [21, 22]. Section 4 includes the supersymmetry analysis in terms of generalized Spin(7) structures. The final section includes some discussion of future directions. To improve the presentation of the paper, almost all of the technical details of the supersymmetry analysis have been relegated to the appendices.

2. G-structures and intrinsic torsion

In this section we give a brief review of G-structures and intrinsic torsion with emphasis on the points relevant to our case.

Quite generally, the requirement of supersymmetry for backgrounds of the form $\mathcal{M} \times_w X$, where $\mathcal{M}$ is maximally-symmetric, implies the existence of a nowhere-vanishing spinor ξ satisfying a Killing equation

$$\nabla_m \xi = G_m \xi\,, \tag{2.1}$$

where ∇_m is the Levi-Civita connection and G_m is a vector field on X taking values in the Clifford algebra Cliff(d), $\;d := \dim_{\mathbb{R}}(X)$. The exact expression depending on the supergravity under consideration, G_m is determined by the fluxes (i.e. antisymmetric tensor fields) on X and is generally nonzero. If G_m vanishes identically, X is a special-holonomy manifold.

Typically, the existence of ξ implies the reduction of the structure group to a subgroup $G \subset$ Spin(d) and the manifold X admits a G-structure. The latter is characterized by the intrinsic torsion, which is a measure of the failure of ξ to be covariantly constant with respect to the connection associated with the metric induced by the G-structure. From what has been just said, it is clear that the intrinsic torsion could be read off of G_m in equation (2.1) above. Hence the result that the intrinsic torsion can be expressed in terms of the fluxes. Schematically:

$$\text{flux} \longrightarrow G_m \longrightarrow \text{intrinsic torsion} \ . \tag{2.2}$$

Typically, the Killing spinor ξ gives rise to certain G-invariant forms (Φ), constructed out of ξ as spinor bilinears. It is an important result that the intrinsic torsion (ω) can also be read off of the exterior derivatives of these forms:

$$\nabla\xi \longleftrightarrow \omega \longleftrightarrow d\Phi \ . \tag{2.3}$$

Having a dictionary between these two alternative descriptions is very useful in practice, as it allows us to express the supersymmetry equation (2.1) in purely algebraic form. We will see how this works explicitly in section 3.1.

The intrinsic torsion can be decomposed in terms of irreducible G-modules: $\omega \in \Lambda^1 \otimes \mathfrak{g}^\perp$, where $\mathfrak{g}^\perp$ is the complement of $\mathfrak{g} := \mathrm{Lie}(G)$ inside $spin(d)$. Special classes of manifolds arise when some of these modules vanish; for example when all the modules vanish the manifold is special-holonomy. The construction/classification of manifolds according to their intrinsic torsion is a difficult problem which still remains largely open. There is of course great interest from the physics point-of-view because of the connection to flux compactifications. In some cases it is not known if examples of manifolds exist in all classes of possible combinations of nonzero modules.

2.1 Spin(7) structure

Let us now see how the general discussion of G-structures applies to the case where X is an eight-manifold. The main result explained here is that the existence of a nowhere-vanishing Majorana spinor on X, $\xi = \xi^+ \oplus \xi^-$ where ξ^+ (ξ^-) is of positive (negative) chirality, induces the reduction of the structure group of the associated nine-manifold $Y := X \times S^1$, where S^1 is a circle of constant radius, to Spin(7).

A Spin(7) structure on X is a principal sub-bundle of the frame bundle over X, with fiber the subgroup Spin(7) of $GL(8, \mathbb{R})$. We can give alternative description of the Spin(7) structure as follows [23]: let $x^1, \ldots, x^8$ be the coordinates of $\mathbb{R}^8$. The self-dual four-form

$$\begin{aligned}\Phi_0^+ := &e^{1234} + e^{1256} + e^{1278} + e^{1357} - e^{1368} - e^{1458} - e^{1467} \\ &- e^{2358} - e^{2367} - e^{2457} + e^{2468} + e^{3456} + e^{3478} + e^{5678} \,,\end{aligned} \tag{2.4}$$

where e^{ijkl} denotes $dx^i \wedge dx^j \wedge dx^k \wedge dx^l$, is fixed by $\mathrm{Spin}(7) \subset GL(8, \mathbb{R})$. At each point $p \in X$ let us define $\mathcal{A}_pX$ to be the subset of four-forms $\Phi_p \in \Lambda^4 T_p^* X$ for which there exists an isomorphism between T_pX and $\mathbb{R}^8$ such that Φ_p is identified with Φ_0^+. It follows that $\mathcal{A}_pX$ is isomorphic to $GL(8, \mathbb{R})/\,\mathrm{Spin}(7)$. Let $\mathcal{A}X$ be the bundle over X with fiber $\mathcal{A}_pX$ for each $p \in X$. We say that a four-form Φ on X is *admissible* if $\Phi_p \in \mathcal{A}_pX$ for each $p \in X$.

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In other words, admissible forms are those that can be 'reached' from Φ_0^+, at each point in X. It follows that there is a 1-1 correspondence between Spin(7) structures and admissible four-forms $\Phi \in \mathcal{A}X$.

The isotropy group of a nonzero Majorana-Weyl spinor of Spin(8), is Spin(7). This simply follows from the fact that under Spin(7) $\subset$ Spin(8) the chiral spinor representation of Spin(8) decomposes as $\mathbf{8} \to \mathbf{7} + \mathbf{1}$, i.e. there is a singlet in the decomposition. Hence a nowhere-vanishing Majorana-Weyl spinor of Spin(8) induces a reduction of the structure group of X to Spin(7). An equivalent way to understand the reduction of the structure group is by noting that there is a nowhere-vanishing self-dual four-form which can be constructed as a bilinear of the chiral spinor.

Let $I_\pm$ be the isotropy groups of $\xi^\pm$. The isotropy group $I_+ \cap I_-$ of $\xi = \xi^+ \oplus \xi^-$ induces a local reduction of the structure group. If both Majorana-Weyl spinors $\xi^\pm$ are nowhere-vanishing, the structure group of X reduces to the common subgroup of the two Spin(7) structures: $\mathrm{Spin}(7)_+ \cap \mathrm{Spin}(7)_- = G_2$. The reduction of the structure group to G_2 can be deduced alternatively from the fact that in this case there exist a nowhere-vanishing vector and a three-form on X, which can be constructed as a bilinears of ξ^+ and ξ^-.

In general, both chiral spinors may have zeros. At a point where ξ^+ (ξ^-) vanishes, I_+ (I_-) is enhanced to Spin(8) and the isotropy group of $\xi = \xi^+ \oplus \xi^-$ is enhanced to $I_+ \cap I_- = \mathrm{Spin}(7)_\mp$. This, however, does not induce a reduction of the structure group to Spin(7) unless, as explained in the previous paragraph, X supports a nowhere-vanishing Majorana-Weyl spinor. Roughly-speaking, the isotropy group of ξ is not, in general, a fixed Spin(7) subgroup of Spin(8) — as is required for a reduction of the structure group: as one moves around in X, $I_+ \cap I_-$ 'rotates' inside Spin(8).

Nevertheless, it is possible to translate this situation to an honest Spin(7) reduction of the structure group not of X itself, but of an associated nine-manifold. First, let us recall some useful facts about *Pin* vs *Spin* groups. The reader can consult, e.g., [24] for more details. The group $Pin(n)$ sits inside $\mathrm{Cliff}(n)$, therefore the irreducible representations of $\mathrm{Cliff}(n)$ restrict to representations of $Pin(n)$ — which actually turn out to be irreducible as well. In particular, the real irreducible representation of $Pin(8)$ is the restriction of the real irreducible representation of $\mathrm{Cliff}(8)$ — the sixteen-dimensional Majorana spinor in eight dimensions. Similarly, since $\mathrm{Spin}(n)$ sits inside $\mathrm{Cliff}_0(n)$ (the even part of the Clifford algebra), the irredicible representations of $\mathrm{Cliff}_0(n)$ restrict to representations of $\mathrm{Spin}(n)$ — which also turn out to be irreducible. Furthermore, there is an important isomorphism:

$$\mathrm{Cliff}(n) \cong \mathrm{Cliff}_0(n+1) \ . \tag{2.5}$$

Coming back to physics: we are considering M-theory compactifications on an eight-manifold X. Supersymmetry imposes the existence of a nowhere-vanishing real pinor ξ on X. Tensoring X by a constant circle S^1, we obtain a compact nine-dimensional product manifold $Y := X \times S^1$. As can be seen from (2.5), the nowhere-vanishing pinor ξ on X lifts to a nowhere-vanishing spinor on Y in the real, sixteen-dimensional irreducible representation of Spin(9). Of course ξ, thought of as a spinor on Y, does not depend on the co-ordinate of S^1. It is well-known that the existence of a nowhere-vanishing spinor on a nine-manifold Y induces the reduction of the structure group of Y to Spin(7).[1]

Alternatively, the reduction of the structure group of Y can be understood as follows: Since ξ is nowhere-vanishing, the Majorana spinors $\epsilon^{\pm} := \frac{1}{\sqrt{2}}(\xi^+ \pm \xi^-)$ (the normalization is for later convenience) are also nowhere-vanishing and each of them induces a reduction of the structure group of Y to Spin(7). At the points in Y where $\epsilon^{\pm}$ are not parallel, the isotropy group of $\epsilon^{\pm}$ is reduced to $G_2 = \mathrm{Spin}(7) \cap \mathrm{Spin}(7)$. However, ϵ^+ becomes parallel to ϵ^- precisely at the points where either of $\xi^{\pm}$ vanishes. At these points the isotropy group is enhanced to Spin(7). This point-of-view is better suited for the description in terms of *generalized* Spin(7) structures, which we introduce in section 4.1 below.

The topological obstruction to the existence of a nowhere-vanishing Majorana-Weyl spinor on X is known [25, 26]: it is equivalent to the condition that the Euler number of X be given by

$$\chi(X) = \pm\frac{1}{2}\int_X \left(p_2 - \frac{1}{4}p_1^2\right), \tag{2.6}$$

where $p_{1,2}$ are the first and second Pontrjagin forms. The sign on the right-hand side of the equation above depends on the chirality of the nowhere-vanishing spinor. As is it follows immediately from (2.6), requiring that nowhere-vanishing Majorana-Weyl spinors of *both* chiralities exist, leads to the condition that the Euler number of X vanishes. This, of course, is exactly the topological condition for the existence of a nowhere-vanishing vector field on X. Indeed, a vector field can be constructed as a bilinear of ξ^+ and ξ^- and, interestingly, equations (2.6) can be related to the condition $\chi(X) = 0$ by a certain triality rotation [26]. As we have seen, in this case there is a reduction of the structure group to G_2. In the generic case, although there is a nowhere-vanishing Majorana spinor $\xi = \xi^+ \oplus \xi^-$ on X, both $\xi^{\pm}$ may have zeros; hence X need *not* satisfy equation (2.6).

3. $\mathcal{N} = 1$ supersymmetry

The starting point of our analysis is the supersymmetry equations given in (3.5)–(3.8) below. We will now describe the Anszätze leading up to these equations, as well as some basic background on eleven-dimensional supergravity in order to establish conventions. This brief review follows [19].

The field content of eleven-dimensional supergravity consists of a metric, a Majorana vector-spinor (gravitino) and a four-form field strength G. We shall consider eleven-dimensional M-theory backgrounds of the form of a warped product $\mathcal{M} \times_w X$, where X is an eight-manifold and $\mathcal{M}$ is three-dimensional Minkowski or AdS space. Explicitly, the metric Ansatz reads

$$ds_{11}^2 = e^{2\Delta}(ds_3^2 + g_{mn}dx^m dx^n)\,, \tag{3.1}$$

[1]Recall that Spin(9) acts transitively on the unit sphere in sixteen dimensions, $S^{15} \cong \mathrm{Spin}(9)/\mathrm{Spin}(7)$, and that a real, sixteen-dimensional unit spinor on Y implies the existence of a global section of the sphere bundle over Y with fiber S^{15}.

where $e^{2\Delta}$ is the warp factor and ds_3^2 is the metric on $\mathcal{M}$. For the convenience of the reader, we follow the notation of [19]. The most general four-form flux Ansatz respecting three-dimensional covariance reads

$$G = e^{3\Delta}(F + Vol_3 \wedge f)\,, \tag{3.2}$$

where Vol_3 is the volume element along the noncompact directions and f (F) is a one-form (four-form) on X. Finally, the eleven-dimensional supersymmetry parameter ζ splits into a direct product of a Majorana spinor ψ on $\mathcal{M}$ and a spinor $\xi = \xi^+ \oplus \xi^-$ on X:

$$\zeta = e^{-\frac{\Delta}{2}}\psi \otimes (\xi^+ \oplus \xi^-)\,. \tag{3.3}$$

More precisely, ξ is a section of the real spin sub-bundle $S^+_{\mathbb{R}} \oplus S^-_{\mathbb{R}}$ on X, where $S^+ \oplus S^-$ is the spin bundle on X and $S^\pm = S^\pm_{\mathbb{R}} \otimes \mathbb{C}$. Furthermore, since $\mathcal{M}$ is Minkowski or AdS, ψ is constrained to satisfy

$$\nabla_\mu \psi + m\gamma_\mu \psi = 0\,, \tag{3.4}$$

where m is a massive parameter proportional to the inverse radius of $\mathcal{M}$. Substituting our Ansätze into the supersymmetry transformations of eleven-dimensional supergravity, we arrive at the following equations.

Internal gravitino:

$$0 = \nabla_m \xi^+ + \frac{1}{24}F_{mpqr}\gamma^{pqr}\xi^- - \frac{1}{4}f_n\gamma^n{}_m\xi^+ - m\gamma_m\xi^- \tag{3.5}$$
$$0 = \nabla_m \xi^- + \frac{1}{24}F_{mpqr}\gamma^{pqr}\xi^+ + \frac{1}{4}f_n\gamma^n{}_m\xi^- + m\gamma_m\xi^+\,. \tag{3.6}$$

External gravitino:

$$0 = \frac{1}{2}\gamma^m\partial_m\Delta\xi^+ - \frac{1}{288}F_{mpqr}\gamma^{mpqr}\xi^- - \frac{1}{6}\gamma^n f_n\xi^+ + m\xi^- \tag{3.7}$$
$$0 = \frac{1}{2}\gamma^m\partial_m\Delta\xi^- - \frac{1}{288}F_{mpqr}\gamma^{mpqr}\xi^+ + \frac{1}{6}\gamma^n f_n\xi^- - m\xi^+\,. \tag{3.8}$$

We have thus rewritten the eleven-dimensional supersymmetry transformations purely in terms of fields on X.

In addition to the supersymmetry equations, a solution of eleven-dimensional supergravity should satisfy the Bianchi identities and the equations-of-motion. It can be shown that these take the form[2]

$$\begin{aligned} 0 &= d(e^{3\Delta}F) \\ 0 &= e^{-6\Delta}d(e^{6\Delta}\star f) - \frac{1}{2}F\wedge F \\ 0 &= e^{-6\Delta}d(e^{6\Delta}\star F) - f\wedge F\,. \end{aligned} \tag{3.9}$$

[2]The analysis of [19] suggests that the last line in (3.9) may be redundant, i.e. it may follow from the supersymmetry equations and the remaining equations in (3.9). We thank D. Martelli for discussions on this point.

One can show that, under a certain mild condition which is satisfied for the backgrounds considered in this paper, the supersymmetry equations together with the Bianchi identities and equations-of-motion (3.9) imply the Einstein equations [27]. Similar integrability statements also hold for IIA [28] and IIB [29] supergravities.

Note that setting $m = 0$ excludes any solutions with nonzero fluxes if X is smooth and compact. As noted in [19], this can be seen immediately from the scalar part of the Einstein equation:

$$e^{-9\Delta}\Box e^{9\Delta} - \frac{3}{2}|F|^2 - 3|f|^2 + 72m^2 = 0 \; . \tag{3.10}$$

Integrating by parts gives f, $F = 0$. This no-go 'theorem' can be evaded by allowing the equations-of-motion and/or Bianchi identities to be modified, e.g. by introducing source terms or higher-order curvature corrections.

A well-known higher-order correction is the one related to the M five-brane anomaly [30]

$$d \star G + \frac{1}{2} G \wedge G = \beta X_8 \, , \tag{3.11}$$

where β is a constant of order l^6_{Planck} and X_8 is proportional to the same combination of Pontrjagin forms appearing on the right-hand side of (2.6). However, generally it is *inconsistent* to only include the correction (3.11) without considering the corresponding order-l^6_{Planck} corrections to the supersymmetry equations. The latter corrections are, unfortunately, unknown to date.[3]

The implications of the supersymmetry equations above are examined in section 4.1 from the point of view of generalized Spin(7) structures. The more conventional approach is pursued in section 3.1. Before we close this section, let us make an observation which will be important in the following: as we explain in appendix D, it follows from (3.5), (3.6) that the Majorana spinor ξ has constant norm, which we can normalize to unity without loss of generality:

$$|\xi|^2 = |\xi^+|^2 + |\xi^-|^2 = 1 \; . \tag{3.12}$$

This equation was first noticed in [19].

3.1 Analysis

It follows from equation (3.12) that at each point p in X at least one of $\xi^\pm$, let us say ξ^+ for concreteness, is non-vanishing. In an open set around p, we can parameterize:

$$\begin{aligned} \xi^+ &= \frac{1}{\sqrt{1+L^2}}\eta \\ \xi^- &= \frac{L_m}{\sqrt{1+L^2}}\gamma^m\eta \, , \end{aligned} \tag{3.13}$$

[3] In certain cases, e.g. compactification on 'large' eight-manifolds, it is in fact consistent to ignore all higher-order corrections except for the one in (3.11), see [31] for a detailed argument.

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where η has unit norm: $|\eta|^2 = 1$. Note that the one-form L can be thought of as a ξ^+, ξ^- bilinear. Moreover, we can define a self-dual four-form Φ as an η-bilinear via[4]

$$\Phi_{mpqr} := \eta\gamma_{mpqr}\eta \ . \tag{3.14}$$

As was explained in detail in section 2.1, the existence of the four-form defined in (3.14) above induces a Spin(7)-structure on X; therefore, one can decompose all fields in terms of irreducible representations of Spin(7). As was mentioned in section 2, it is very useful to be able to translate back and forth between the spinor and the G-structure language; in this way the supersymmetry equations can be expressed as a set of purely algebraic relations. For the case at hand, the schematic equation (2.3) is nothing but the statement that the following two equations are equivalent:

$$\boxed{\begin{aligned} \partial_{[m}\Phi_{pqrs]} &= -8\Phi_{[mpqr}\omega^1_{s]} - \frac{4}{15}\varepsilon_{mpqrs}{}^{ijk}\omega^2_{ijk} \\ \nabla_m\eta &= \Big\{\omega^1_n\gamma^n{}_m + \omega^2_{mpq}\gamma^{pq}\Big\}\eta\,, \end{aligned}} \tag{3.15}$$

where ω^1 transforms in the **8** of Spin(7) while ω^2 transforms in the **48**. Note that $d\Phi$ being a five-form it transforms in the $\mathbf{8} \oplus \mathbf{48}$ of Spin(7), hence the decomposition on the right-hand side of the first line of (3.15). $\omega^{1,2}$ generate the two modules of the intrinsic torsion of a manifold of Spin(7) structure [32]. The equivalence of the two equations in (3.15), is proven in appendix E.

Skipping all the details of the derivation, which can be found in appendix D, the supersymmetry conditions are equivalent to the following equations: the one-form L is constrained to satisfy

$$\boxed{\begin{aligned} d\Big(e^{3\Delta}\frac{L}{1+L^2}\Big) &= 0 \\ e^{-12\Delta}\star d\star\Big(e^{12\Delta}\frac{L}{1+L^2}\Big) - 4m\frac{1-L^2}{1+L^2} &= 0\,, \end{aligned}} \tag{3.16}$$

where here and in the remainder of this paper the Hodge star is taken with respect to the internal eight-dimensional space. Moreover, all flux components except for the **27**

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[4]In our conventions the real spinor η satisfies $\eta^\dagger = \eta^{\mathrm{Tr}}C$, where C is the charge-conjugation matrix. We use C to raise/lower indices on the gamma-matrices, so that the notation $\eta\gamma_m\eta$ is a shorthand for $\eta^{\mathrm{Tr}}(C\gamma_m)\eta$, etc.

component of F are solved for in terms of L and the warp factor:

$$\boxed{\begin{aligned} f &= e^{-3\Delta} d\Big(e^{3\Delta}\frac{1-L^2}{1+L^2}\Big) + 8m\frac{L}{1+L^2} \\ \frac{1}{12}F^{\mathbf{1}} &= e^{-3\Delta}L^i\partial_i\Big(\frac{e^{3\Delta}}{1+L^2}\Big) - m\frac{3-L^2}{1+L^2} \\ \frac{1}{96}F^{\mathbf{7}}_{rs} &= -e^{-3\Delta}(P^{\mathbf{7}})_{rs}^{pq}L_p\partial_q\Big(\frac{e^{3\Delta}}{1+L^2}\Big) \\ \frac{1}{24}F^{\mathbf{35}}_{mn} &= -\nabla_{(m}L_{n)} - \frac{1}{4}\Phi_{(m}{}^{ijk}(L\otimes F^{\mathbf{27}})^{\mathbf{48}}_{n)ij}L_k + \frac{3}{7(1+L^2)^2}\Big(L_mL_n + \frac{L^2}{6}g_{mn}\Big)L^i\nabla_iL^2 \\ &\quad -\frac{9}{7(1+L^2)}\Big\{L_mL_n + \frac{7+8L^2}{6}g_{mn}\Big\}L^i\nabla_i\Delta + \frac{1}{(1+L^2)^2}L_{(m}\partial_{n)}L^2 \\ &\quad +\frac{3L^2}{1+L^2}L_{(m}\partial_{n)}\Delta + \frac{m}{14(1+L^2)}\Big\{8(L^2-3)L_mL_n + (7+3L^2-8L^4)g_{mn}\Big\}\,, \end{aligned}} \tag{3.17}$$

where the explicit decomposition of F in terms of irreducible representations of Spin(7) and the explanation of the definitions entering the equations above, are given in the appendix. As noted in the introduction, in open sets where neither of $\xi^{\pm}$ vanishes, the above equations should reduce to the corresponding formulæ given in [19], to the extent they overlap (some of the flux components were not given explicitly in [19]). Finally, the intrinsic torsion is determined via

$$\boxed{\begin{aligned} \omega^1_m &= \frac{m}{2}L_m + \frac{3}{4}\partial_m\Delta + \frac{1}{168}(L_mF^{\mathbf{1}} - L^iF^{\mathbf{7}}_{im}) \\ \omega^2_{mpq} &= \frac{1}{192}(L\otimes F^{\mathbf{7}})^{\mathbf{48}}_{mpq} + \frac{1}{4}(L\otimes F^{\mathbf{27}})^{\mathbf{48}}_{mpq}\,. \end{aligned}} \tag{3.18}$$

3.2 Small-flux approximation

A special solution to the supersymmetry equations (3.5)–(3.8) the Bianchi identities and the equations-of-motion (3.9), is obtained when the warp factor and all flux vanishes (Δ, f, $F=0$), $\mathcal{M}$ is three-dimensional Minkowski space ($m=0$) and X is a manifold of Spin(7) holonomy ($\omega^{1,2}=0$). In this section we would like to perform a small perturbation around the special-holonomy solution; this amounts to a small-flux approximation. Note that this is an expansion around the point where the $G_2 = \mathrm{Spin}(7)_+ \cap \mathrm{Spin}(7)_-$ structure breaks down, and so it cannot be described by the formalism of [19].

For each field S, let us make a perturbative expansion

$$S = \sum_{n=0} S^{(n)}\varepsilon^n\,, \tag{3.19}$$

where ε is a small parameter, so that the special-holonomy solution is recovered in the $\varepsilon \to 0$ limit. Equations (3.17) determine the flux components:

$$\begin{aligned} F^{\mathbf{1}} &= -36m^{(1)}\varepsilon + \mathcal{O}(\varepsilon^2) \\ F^{\mathbf{7}}_{mn} &= \mathcal{O}(\varepsilon^2) \\ F^{\mathbf{35}}_{mn} &= \Big(-24\nabla_{(m}L^{(1)}_{n)} + 12g_{mn}m^{(1)}\Big)\varepsilon + \mathcal{O}(\varepsilon^2)\,, \end{aligned} \tag{3.20}$$

where the metric and the Levi-Civita connection above are those of the unperturbed special-holonomy solution. The **27** component of the flux is of order $\mathcal{O}(\varepsilon)$, but is otherwise unrestricted by the supersymmetry equations. Moreover, equations (3.18) give

$$\begin{aligned} \omega^1_m &= \frac{3}{4}\partial_m \Delta^{(1)} + \mathcal{O}(\varepsilon^2) \\ \omega^2_{mpq} &= \mathcal{O}(\varepsilon^2) \ . \end{aligned} \tag{3.21}$$

It follows from the form of the intrinsic torsion (see e.g. [33]) that to order $\mathcal{O}(\varepsilon^2)$ the eight-manifold is conformally special-holonomy. Finally, to order $\mathcal{O}(\varepsilon^2)$, equations (3.16) are equivalent to

$$\begin{aligned} \nabla^m L^{(1)}_m &= 4m^{(1)} \\ \nabla_{[m} L^{(1)}_{n]} &= 0 \, , \end{aligned} \tag{3.22}$$

where again the Levi-Civita connection above is that of the special-holonomy metric. Integrating the first line by parts in the case where X is smooth and compact, we conclude that $m^{(1)} = 0$. By the same reasoning, it can be seen by induction that m vanishes to all orders in ε. It follows that, as noted in section 3, all flux vanishes and we get back the special-holonomy solution.

Nontrivial solutions can be obtained if X is noncompact. In this case equations (3.22) are solved for

$$\begin{aligned} L^{(1)}_m &= \partial_m \phi \\ \Box \phi &= 4m^{(1)} \, , \end{aligned} \tag{3.23}$$

where ϕ is a scalar on X with dimensions of length and the box operator is taken with respect to the unperturbed special-holonomy metric. The Bianchi identities and the equations-of-motion impose the conditions that $\Delta^{(1)}$ should be harmonic and that $F^{\mathbf{27}}$ should be closed. Neglecting corrections of order $\mathcal{O}(\varepsilon^2)$, it can be shown that there are no further conditions.[5]

As an explicit example, let us consider a small perturbation around the noncompact Spin(7)-holonomy metric of reference [21, 22]:

$$ds^2 = \left(1 - \left(\frac{l}{r}\right)^{10/3}\right)^{-1} dr^2 + \frac{9}{100} r^2 \left(1 - \left(\frac{l}{r}\right)^{10/3}\right)(\sigma_i - A^i)^2 + \frac{9}{20} r^2 d\Omega_4^2 \, , \tag{3.24}$$

[5] To arrive at this result the only nontrivial step is to prove that $I_{m_1 \ldots m_5} := \Phi_{[m_1 \ldots m_3}{}^p R_{m_4 m_5], p}{}^q L_q$ vanishes, where the Riemann tensor is with respect to the special-holonomy connection. This can be seen as follows: for fixed m, n, $R_{mn,pq}$ can be viewed as an antisymmetric matrix with indices p, q; i.e. it transforms in the $\mathbf{21} + \mathbf{7}$ of *spin*(7). However, since the spinor η is parallel with respect to the connection, it follows that $R_{mn,pq}\gamma^{pq}\eta = 0$ and hence the **21** component is projected out while the **7** component is set to zero (cf. equation (B.6)). I.e. for fixed m, n (or for fixed p, q, thanks to the symmetry properties of the Riemann tensor) $R_{mn,pq}$ transforms in the **21** of *spin*(7). Furthermore, as follows from the symmetry of the free indices and the previous discussion, $R_{mn,p}{}^q L_q$ transforms in the $\mathbf{21} \otimes \mathbf{8} = \mathbf{8} \oplus \mathbf{48} \oplus \mathbf{112}$. On the other hand, $I_{m_1 \ldots m_5}$ is a five-form and hence it transforms in the $\mathbf{8} \oplus \mathbf{48}$ of *spin*(7); therefore the **112** representation is projected out. The remaining $\mathbf{8} \oplus \mathbf{48}$ representations are generated by $R_{[mnp]}{}^q L_q$, which vanishes by virtue of the symmetries of the Riemann tensor, and $g_{mn} R_p{}^q L_q$, which vanishes by virtue of the fact that every manifold of special holonomy is Ricci-flat. It follows that $I_{m_1 \ldots m_5}$ vanishes.

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where $l \le r < \infty$, $d\Omega_4^2$ is the metric of the unit four-sphere S^4, $\{\sigma_i;\ i = 1,2,3\}$ are left-invariant SU(2) one-forms and A^i is the connection of a Yang-Mills instanton on S^4. Moreover, let us assume that $\phi = \phi(r)$. For $r \gg l$, the solution to equation (3.23) behaves as

$$\phi \sim Q_1 + Q_2 \frac{1}{\delta^2} + \mathcal{O}(\delta)\,,$$

where $\delta := l/r$ and $Q_{1,2}$ are constants (Q_2 depends on m). On the other hand, when r approaches l we have

$$\phi \sim Q_1' + Q_2'\Big(\frac{1}{\delta} + \frac{8}{3}\log(\delta)\Big) + \mathcal{O}(\delta)\,,$$

where $Q_{1,2}'$ are constants (Q_2' depends on m) and $\delta := r/l - 1$. In conclusion: for $m \neq 0$ (i.e. for $\mathcal{M}$ AdS), $L^2 \sim |\partial\phi|^2$ blows up near l, ∞, which is contrary to our assumption that L is perturbatively small. Hence, the solution can only be trusted for intermediate distances $1/\varepsilon \gg r/l \gg \varepsilon$. Note however that for $m = 0$ (i.e. for $\mathcal{M}$ Minkowski) Q_2 vanishes and the solution to (3.23) is regular for large distances $r/l \gg \varepsilon$.

4. Generalized G-structures

In this section, after some preliminaries, we give the definition of generalized Spin(7) structures in nine dimensions and explain how they arise naturally in the context of supersymmetric M-theory compactifications on eight-manifolds. Generalized G-structures were first introduced in [8]. Generalized Spin(7) structures in eight dimensions were first examined by F. Witt in [15].

Consider the direct sum of the tangent and cotangent bundle $T \oplus T^*$ of a d-dimensional manifold X. There is a natural action of $T \oplus T^*$ on forms, whereby every vector acts by contraction and every form by exterior multiplication. Explicitly: if V is a vector on X and U, Ω are forms, we define

$$(V + U)\cdot\Omega = \iota_V\Omega + U \wedge \Omega\ . \tag{4.1}$$

As this action squares to the identity, there is an associated Clifford algebra Cliff$(T \oplus T^*)$ and an induced isomorphism

$$\mathrm{Cliff}(T \oplus T^*) \approx \mathrm{End}(\Lambda^*)\ . \tag{4.2}$$

A basis of the Clifford algebra on $T \oplus T^*$

$$\{\gamma^m, \gamma^n\} = 0; \qquad \{\gamma_m, \gamma_n\} = 0; \qquad \{\gamma^m, \gamma_n\} = \delta^m_n \tag{4.3}$$

is given explicitly by $\gamma^m := dx^m\wedge$, $\gamma_n := \iota_n$. It follows from the isomorphism (4.2) above that (sums of) forms on X can be identified with spinors of $T \oplus T^*$. Moreover, the latter can be thought of as bispinors on X. We thus obtain

$$\text{forms} \longleftrightarrow \text{spinors on } T \oplus T^* \longleftrightarrow \text{bispinors on } X\ . \tag{4.4}$$

The identification of sums of forms with bispinors is, of course, explicitly realized by Fierzing.

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4.1 Generalized Spin(7) structures

Coming back to our eight-dimensional case, we define the following bispinors

$$\Phi^{\pm} := \xi^{\pm} \otimes \xi^{\pm} = \frac{1}{8} P_{\pm} \Big\{ \Phi^{\pm} + \frac{1}{2 \cdot 4!} \Phi^{\pm}_{m_1 \dots m_4} \gamma^{m_1 \dots m_4} + \frac{1}{8!} \Phi^{\pm}_{m_1 \dots m_8} \gamma^{m_1 \dots m_8} \Big\} , \tag{4.5}$$

where $P_{\pm} := \frac{1}{2}(1 \pm \gamma_9)$ is the chirality projector and $\Phi^{\pm}_{m_1 \dots m_p} := \xi^{\pm} \gamma_{m_1 \dots m_p} \xi^{\pm}$. By a slight abuse of notation, we use the same letter to denote both the bispinor and the associated forms. It should be clear from the context which one is meant in each case. It will prove more convenient to work with the linear combinations: $\Psi^{\pm} := \frac{1}{2}(\Phi^{+} \pm \Phi^{-})$. Moreover we define

$$\widehat{\Psi} := \xi^{+} \otimes \xi^{-} = \frac{1}{8} P_{+} \sum_{p = \text{odd}} \frac{1}{p!} \widehat{\Psi}_{m_1 \dots m_p} \gamma^{m_1 \dots m_p} , \tag{4.6}$$

where $\widehat{\Psi}_{m_1 \dots m_p} := \xi^{+} \gamma_{m_1 \dots m_p} \xi^{-}$. In the following we will find it useful to define the combinations (cf. also equations (F.7), (F.8) of appendix F): $\widehat{\Psi}^{\pm} := \frac{1}{2}(\widehat{\Psi} \pm \star \widehat{\Psi})$.

The Majorana spinors (real pinors) $\epsilon^{\pm} := \frac{1}{\sqrt{2}}(\xi^{+} \pm \xi^{-})$ are nowhere-vanishing on X, since $2|\epsilon|^2 = |\xi^{+}|^2 + |\xi^{-}|^2 = 1$. Hence any bipinors constructed out of the ϵ's are also nowhere-vanishing. Setting

$$\begin{aligned} \rho^{\pm} &:= \epsilon^{\pm} \otimes \epsilon^{\pm} = \Psi^{+} \pm \widehat{\Psi}^{-} \\ \widehat{\rho}^{\pm} &:= \epsilon^{\pm} \otimes \epsilon^{\mp} = \Psi^{-} \mp \widehat{\Psi}^{+} , \end{aligned} \tag{4.7}$$

we note that $\Psi^{\pm}$, $\widehat{\Psi}^{\pm}$ can be expressed as linear combinations of $\rho, \widehat{\rho}$. As discussed in detail in section 2.1, these bipinors on X lift to bispinors on $Y = X \times S^1$. We shall call the pair $(\rho,\ \widehat{\rho})$ a *generalized* Spin(7) *structure on* Y. Note that $(\rho,\ \widehat{\rho})$ induce a reduction of the structure group Spin(9, 9) of $TY \oplus T^*Y$ to Spin(7) $\times$ Spin(7). This follows from the fact that $\epsilon^{\pm}$ are nowhere-vanishing and therefore each of them induces a reduction of the structure group of Y to Spin(7), as explained in 2.1.

As we show in appendix F, $\mathcal{N} = 1$ supersymmetry implies that the generalized Spin(7) structure, or equivalently the bispinors $\Psi^{\pm}$, $\widehat{\Psi}^{\pm}$, satisfy the following differential equations

$$\boxed{\begin{aligned} 0 &= d\Psi^{+} + F \wedge \widehat{\Psi}^{-} \\ 0 &= e^{-3\Delta} d(e^{3\Delta} \Psi^{-}) + \star F \wedge \widehat{\Psi}^{-} - f \wedge \Psi^{+} + 4m \widehat{\Psi}^{-} \\ 0 &= e^{-3\Delta/2} d(e^{3\Delta/2} \widehat{\Psi}^{-}) \\ 0 &= e^{-3\Delta/2} d(e^{3\Delta/2} \widehat{\Psi}^{+}) + 2(\star F \wedge \Psi^{+} - F \wedge \Psi^{-}) + 8m \Psi^{+} . \end{aligned}} \tag{4.8}$$

In the terminology of [4, 7], the equations (4.8) above are the 'form picture' of the $\mathcal{N} = 1$ supersymmetry equations given in the 'spinor picture' in (3.5)–(3.8). Note that, as is easy to show, the integrability of (4.8) follows from the equations-of-motion and the Bianchi identities.

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4.2 Reduction to seven dimensions

In the case where X is of the form $Z \times S^1$ and assuming no fields depend on the coordinate of S^1, we can perform a reduction to seven dimensions –upon which $\xi^\pm$ and $\epsilon^\pm$ transform in the $\mathbf{8}$ of Spin(7). Since as we noted above $\epsilon^\pm$ are nowhere-vanishing, each of them gives rise to a $G_{2\pm} \subset \mathrm{Spin}(7)$ structure on Z. Indeed, G_2 is the isotropy group of a fundamental spinor inside Spin(7). Alternatively, this can be seen by noting that under $G_2 \subset \mathrm{Spin}(7)$ the fundamental spinor representation decomposes as $\mathbf{8} \rightarrow \mathbf{7} + \mathbf{1}$, i.e. there is a singlet in the decomposition. If $\epsilon^\pm$ are nowhere parallel (equivalently: if $\xi^\pm$ are nowhere-vanishing), there is a further reduction of the structure group of Z to the common subgroup $G_{2+} \cap G_{2-} = \mathrm{SU}(3)$. In the generic case there are points in Z where $\epsilon^\pm$ become parallel. At these points the SU(3) structure is enhanced to $G_{2+} \cap G_{2-} = G_2$. This situation is best described in the language of generalized G_2 structures in seven dimensions [4, 7].

5. Conclusions

We have presented a formalism for supersymmetric M-theory compactifications on eight-manifolds X, based on the group Spin(7). This is the most suitable language in which to describe compactifications on eight-manifolds of Spin(7) structure, and/or small-flux perturbations around compactifications on manifolds of Spin(7) holonomy. Although supersymmetry does not, in general, imply the reduction of the structure group of X itself, our analysis leads naturally to the emergence of a nine-dimensional manifold $Y = X \times S^1$ whose structure group is reduced to Spin(7). This is reminiscent of the connection between M- and F-theory: in the case where X admits an elliptic fibration, M-theory on X is equivalent to F-theory on $X \times S^1$ [38]. It would be very interesting to explore this similarity further.

In eight dimensions there exists a Hitchin functional involving a certain three-form and its Hodge-dual five-form [9]. This does not seem to be related to the case considered here: we generally do not have any nowhere-vanishing three-form on X. It would be interesting to explore whether such a functional can be constructed using the four-forms $\Phi^\pm$ of section 4.1 and what should be the generalization of stability in this case. A related point is that, as we have already noted in the introduction, in ten dimensions the NS and RR fields play quite different roles with respect to the Hitchin functional construction [7]; this distinction disappears upon lifting to M-theory. It would also be desirable to know whether equations (4.8) can be interpreted as some sort of integrability condition for the generalized Spin(7) structures.

In type II theories the generalized picture provides a natural framework for T-duality in topological models [34 – 36]. It would be interesting to explore this issue in the context corresponding to the setup of this paper, i.e. in the context of an M- or F-theory topological σ-model with target space the eight-manifold X. We expect T-duality to act as a sign flip: $\xi^- \rightarrow -\xi^-$. This amounts to an exchange $\epsilon^+ \leftrightarrow \epsilon^-$ (cf. section 4.1), in analogy to the situation in seven and six dimensions.

At a more mundane level, the present paper opens up the possibility for supersymmetric solutions with all fluxes turned on and with an internal manifold in any of the

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four classes of Spin(7) manifolds. Any explicit examples of such manifolds are, of course, desirable and could provide us with interesting physics; the physics of M theory on eight manifolds is already very rich, even in the case where the internal manifold is special-holonomy. Even in the absence of an explicit metric, the characterization of the most general $\mathcal{N}=1$ backgrounds given in this paper should suffice for a Kaluza-Klein reduction and the derivation of the resulting low-energy supergravity in three dimensions. It will be interesting to pursue this point further.

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A. Gamma-matrix identities in 8d

The gamma matrices in eight dimensions have the following properties

Symmetry:

$$(C\gamma_{(n)})^{\mathrm{Tr}} = (-)^{\frac{1}{2}n(n-1)}C\gamma_{(n)}\,, \tag{A.1}$$

where C is the charge-conjugation matrix.

Hodge-duality:

$$\star\gamma_{(n)} = (-)^{\frac{1}{2}n(n+1)}\gamma_{(8-n)}\gamma_9\,, \tag{A.2}$$

where γ_9 is the chirality matrix.

B. Identities relating to the Spin(7) structure

In this section we give a number of identities which we have used repeatedly in this paper. These can be proved either by Fierzing or by fixing a special basis for the spinor η, as in e.g. [37]. Given a positive-chirality Majorana spinor η of unit norm, we can define a self-dual four-form as in (3.14), which can be seen to satisfy the following identities

$$\begin{aligned}\Phi^{ijkl}\Phi_{ijkm} &= 42\delta^l_m\\ \Phi^{ijkl}\Phi_{ijpq} &= 12\delta^{kl}_{pq} - 4\Phi^{kl}{}_{pq}\\ \Phi^{iklm}\Phi_{ipqr} &= 6\delta^{klm}_{pqr} - 9\Phi^{[kl}{}_{[pq}\delta^{m]}_{r]}\ .\end{aligned} \tag{B.1}$$

Moreover, we have

$$\begin{aligned}\gamma_{ij}\eta &= -\frac{1}{6}\Phi_{ij}{}^{kl}\gamma_{kl}\eta\\ \gamma_{ijk}\eta &= -\Phi_{ijk}{}^{l}\gamma_l\eta\\ \gamma_{ijkl}\eta &= \Phi_{[ijk}{}^{m}\gamma_{l]m}\eta + \Phi_{ijkl}\eta\\ \gamma_{ijklm}\eta &= 5\Phi_{[ijkl}\gamma_{m]}\eta\ .\end{aligned} \tag{B.2}$$

We define the following projectors, acting on a second-rank tensor, onto the **7**, **35** of Spin(7):

$$(P^{\mathbf{7}})_{mn}^{pq} := \frac{1}{4}\Big(\delta^{p}_{[m}\delta^{q}_{n]} - \frac{1}{2}\Phi_{mn}{}^{pq}\Big) \tag{B.3}$$

$$(P^{\mathbf{21}})_{mn}^{pq} := \frac{3}{4}\Big(\delta^{p}_{[m}\delta^{q}_{n]} + \frac{1}{6}\Phi_{mn}{}^{pq}\Big) \tag{B.4}$$

$$(P^{\mathbf{35}})_{mn}^{pq} := \delta^{p}_{(m}\delta^{q}_{n)} - \frac{1}{8}g_{mn}g^{pq} \ . \tag{B.5}$$

A useful identity is

$$(P^{\mathbf{21}})_{rs}^{pq}\gamma_{pq}\eta = 0 \ . \tag{B.6}$$

C. Spin(7) tensor decomposition

Let us decompose F_{mnpq} into irreducible representations

$$F_{mnpq} = F^{\mathbf{1}}_{mnpq} + F^{\mathbf{7}}_{mnpq} + F^{\mathbf{27}}_{mnpq} + F^{\mathbf{35}}_{mnpq} \ . \tag{C.1}$$

Expanding

$$\begin{aligned} F^{\mathbf{1}}_{mnpq} &= \frac{1}{42}\Phi_{mnpq}F^{\mathbf{1}} \\ F^{\mathbf{7}}_{mnpq} &= \frac{1}{24}\Phi_{[mnp}{}^{i}F^{\mathbf{7}}_{q]i} \\ F^{\mathbf{35}}_{mnpq} &= \frac{1}{6}\Phi_{[mnp}{}^{i}F^{\mathbf{35}}_{q]i} \ , \end{aligned} \tag{C.2}$$

where $F^{\mathbf{7}}_{mn}$ is antisymmetric in m, n whereas $F^{\mathbf{35}}_{mn}$ is symmetric and traceless, we obtain

$$F_{ijkp}\Phi^{ijk}{}_{q} = g_{pq}F^{\mathbf{1}} + F^{\mathbf{7}}_{pq} + F^{\mathbf{35}}_{pq} \ . \tag{C.3}$$

In the above we have noted that

$$F^{\mathbf{27}}_{ijkp}\Phi^{ijk}{}_{q} = 0 \ . \tag{C.4}$$

This can be seen immediately as follows: the left-hand side transforms in the $\mathbf{8}\otimes\mathbf{8}$ of Spin(7), however $\mathbf{8}\otimes\mathbf{8} = \mathbf{1}\oplus\mathbf{7}\oplus\mathbf{21}\oplus\mathbf{35}$ and there is no **27** in the decomposition. Note that it follows from decomposition (C.2) that $F^{\mathbf{1}}$, $F^{\mathbf{7}}$ are self-dual while $F^{\mathbf{35}}$ is anti self-dual.

In deriving (D.2) below, we shall need the following decompositions.

$$L_m F^{\mathbf{7}}_{pq} = (P^{\mathbf{7}})_{pq}^{ij}\Big\{(L\otimes F^{\mathbf{7}})^{\mathbf{48}}_{mij} + g_{mi}(L\otimes F^{\mathbf{7}})^{\mathbf{8}}_{j}\Big\} \tag{C.5}$$

$$L_m F^{\mathbf{35}}_{pq} = (P^{\mathbf{35}})_{pq}^{ij}\Big\{\Phi_{mi}{}^{kl}(L\otimes F^{\mathbf{35}})^{\mathbf{48}}_{jkl} + g_{mi}(L\otimes F^{\mathbf{35}})^{\mathbf{8}}_{j}\Big\} + \cdots \ , \tag{C.6}$$

where the ellipses stand for the irreducible representations which drop out of (D.2). These expansions can be inverted to give

$$(L\otimes F^{\mathbf{7}})^{\mathbf{8}}_{m} = \frac{8}{7}L^{i}F^{\mathbf{7}}_{im} \tag{C.7}$$

$$(L\otimes F^{\mathbf{7}})^{\mathbf{48}}_{mpq} = 6\Big(L_{[m}F^{\mathbf{7}}_{pq]} + \frac{1}{7}\Phi_{mpq}{}^{j}L^{i}F^{\mathbf{7}}_{ij}\Big) \tag{C.8}$$

and

$$(L\otimes F^{\mathbf{35}})^{\mathbf{8}}_{m} = \frac{8}{35}L^{i}F^{\mathbf{35}}_{im} \tag{C.9}$$

$$(L\otimes F^{\mathbf{35}})^{\mathbf{48}}_{mpq} = \frac{3}{20}\Big(L_{i}F^{\mathbf{35}}_{j[m}\Phi_{pq]}{}^{ij} - \frac{1}{7}\Phi_{mpq}{}^{j}L^{i}F^{\mathbf{35}}_{ij}\Big)\ . \tag{C.10}$$

The reader can verify that the right-hand sides of (C.8), (C.10) transform in the **48** of Spin(7), as they should. Moreover, note that

$$(L\otimes F^{\mathbf{27}})^{\mathbf{48}}_{mpq} := L^{i}F^{\mathbf{27}}_{impq} \tag{C.11}$$

also transforms in the **48**. Indeed the right-hand side is a three-form and therefore transforms in the $\mathbf{8}^{\mathbf{3}\otimes_{\mathbf{a}}} = \mathbf{8}\oplus\mathbf{48}$. On the other hand, the right-hand side is the product of L, $F^{\mathbf{27}}$, and therefore transforms in the $\mathbf{8}\otimes\mathbf{27} = \mathbf{48}\oplus\mathbf{168}$. It follows that the right-hand side is in the **48** of Spin(7).

D. $\mathcal{N}=1$ supersymmetry

In this appendix we give the details of the derivation of equations (3.12), (3.16)–(3.18). Taking decomposition (C.2) into account, the supersymmetry transformations (3.5)–(3.8) can be seen to be equivalent to the following conditions.

Equation (3.5):

$$0 = \partial_m R - \left(m + \frac{1}{24}F^{\mathbf{1}}\right)L_m + \frac{1}{24}L^{i}(F^{\mathbf{7}}_{im} - F^{\mathbf{35}}_{im}) \tag{D.1}$$

$$0 = (P^{\mathbf{7}})_{rs}^{pq}\Big\{g_{mp}\left(\omega^{1}_{q} - \frac{1}{4}f_q + mL_q\right) - \omega^{2}_{mpq} + \frac{1}{24}(L_p\Phi_q{}^{ijk}F_{mijk} + 6L^{i}F_{impq})\Big\}\ , \tag{D.2}$$

where we have parameterized $\xi^{+} = e^{R}\eta$, $\xi^{-} = e^{R}L_m\gamma^m\eta$. Equation (3.6):

$$\begin{aligned}0 = {}& \nabla_m L_n + \partial_m R L_n + \Phi_n{}^{ijk}\omega^{2}_{mij}L_k - 2\omega^{2}_{mni}L^{i} + g_{mn}\left(L^{i}\omega^{1}_{i} - \frac{1}{4}L^{i}f_i + \frac{1}{24}F^{\mathbf{1}} + m\right)\\ &- L_m\left(\omega^{1}_{n} - \frac{1}{4}f_n\right) - \Phi_{mn}{}^{ij}L_i\left(\omega^{1}_{j} + \frac{1}{4}f_j\right) + \frac{1}{24}(F^{\mathbf{7}}_{mn} + F^{\mathbf{35}}_{mn})\ .\end{aligned} \tag{D.3}$$

Equation (3.7):

$$0 = mL_m + \frac{1}{2}\left(\partial_m\Delta - \frac{1}{3}f_m\right) + \frac{1}{36}L^{i}F^{\mathbf{35}}_{im}\ . \tag{D.4}$$

Equation (3.8):

$$0 = m - \frac{1}{2}L^{i}\left(\partial_i\Delta + \frac{1}{3}f_i\right) + \frac{1}{36}F^{\mathbf{1}} \tag{D.5}$$

$$0 = (P^{\mathbf{7}})_{rs}^{pq}\Big\{L_p\partial_q\Delta + \frac{1}{3}L_pf_q\Big\} + \frac{1}{144}F^{\mathbf{7}}_{rs}\ . \tag{D.6}$$

Before proceding to the derivation of (D.1)–(D.6), let us mention that equation (3.12) is derived as follows: multiplying (D.3) by L^n and using (D.1), we arrive at

$$\partial_m R = -\frac{L^n}{1+L^2}\nabla_m L_n \ . \tag{D.7}$$

Taking into account that $\xi^\pm$ can be rescaled by a real constant without loss of generality, it follows that

$$e^R = \frac{1}{\sqrt{1+L^2}}\,, \tag{D.8}$$

which is equivalent to equation (3.12).

In deriving equations (D.1), (D.2), (D.5), (D.6) we have noted that the equation

$$A_m\eta + B_{m,pq}\gamma^{pq}\eta = 0\,, \tag{D.9}$$

where $(P^{\mathbf{7}})_{pq}^{rs}B_{m,rs} = B_{m,pq}$, is equivalent to $A_m,\ B_{m,pq} = 0$. This can be seen by multiplying on the left by η and $\eta\gamma^{rs}$. Similarly, in deriving (D.3), (D.4) we have noted that the equation

$$A_{m,n}\gamma^n\eta = 0 \tag{D.10}$$

is equivalent to $A_{m,n} = 0$, as can be seen by multiplying on the left by $\eta\gamma_p$.

Equation (D.2) can be used to solve for the intrinsic torsion as in (3.18), by taking into account the following identities

$$0 = (P^{\mathbf{7}})_{rs}^{pq}\Big\{\Phi_{pq}{}^{ab} + 6\delta^a_{[p}\delta^b_{q]}\Big\} \tag{D.11}$$

$$0 = (P^{\mathbf{7}})_{rs}^{pq}\Big\{L_pF^{\mathbf{7}}_{mq} - \frac{1}{4}(L\otimes F^{\mathbf{7}})^{\mathbf{48}}_{mpq} + \frac{5}{7}g_{mp}L^iF^{\mathbf{7}}_{iq}\Big\} \tag{D.12}$$

$$0 = (P^{\mathbf{7}})_{rs}^{pq}\Big\{L^i\Phi_{imp}{}^jF^{\mathbf{7}}_{qj} + \frac{1}{4}(L\otimes F^{\mathbf{7}})^{\mathbf{48}}_{mpq} - \frac{12}{7}g_{mp}L^iF^{\mathbf{7}}_{iq}\Big\} \tag{D.13}$$

$$0 = (P^{\mathbf{7}})_{rs}^{pq}\Big\{L_pF^{\mathbf{35}}_{mq} + 5(L\otimes F^{\mathbf{35}})^{\mathbf{48}}_{mpq} - \frac{1}{7}g_{mp}L^iF^{\mathbf{35}}_{iq}\Big\} \tag{D.14}$$

$$0 = (P^{\mathbf{7}})_{rs}^{pq}\Big\{L^i\Phi_{imp}{}^jF^{\mathbf{35}}_{qj} + 5(L\otimes F^{\mathbf{35}})^{\mathbf{48}}_{mpq} + \frac{6}{7}g_{mp}L^iF^{\mathbf{35}}_{iq}\Big\} \tag{D.15}$$

$$0 = (P^{\mathbf{7}})_{rs}^{pq}\Big\{L_p\Phi_q{}^{ijk}F_{mijk} + L_iF_{jkmp}\Phi_q{}^{ijk} + 4L^iF_{impq}\Big\}\ . \tag{D.16}$$

In order to solve for the $F^{\mathbf{35}}$ component of F, we first define

$$\begin{aligned} f^{\mathbf{35}}_{mn} := F^{\mathbf{35}}_{mn} &+ 2\Big\{L^iF^{\mathbf{35}}_{i(m}L_{n)} - \frac{1}{8}g_{mn}L^iL^jF^{\mathbf{35}}_{ij}\Big\} \\ &+ \frac{6}{7}L^iL^jF^{\mathbf{35}}_{ij}\Big(L_mL_n - \frac{1}{8}g_{mn}L^2\Big)\,, \end{aligned} \tag{D.17}$$

which is the combination that appears in the symmetric, traceless part of (D.3). Note that $f^{\mathbf{35}}$ indeed transforms in the **35** of Spin(7). Equation (D.3) can then be used to solve for $f^{\mathbf{35}}$:

$$\frac{1}{24}f^{\mathbf{35}}_{mn}+\left(\nabla_{(m}L_{n)}-\frac{1}{8}g_{mn}\nabla L\right)+\frac{1}{4}\Phi_{(m}{}^{ijk}(L\otimes F^{\mathbf{27}})^{\mathbf{48}}_{n)ij}L_k$$
$$+\frac{5}{7}\left(L_mL_n-\frac{1}{8}g_{mn}L^2\right)\left\{m(1+\frac{9}{5}L^2)+\frac{9}{5}L^i\partial_i\Delta\right\}=0\,, \tag{D.18}$$

which can be solved for $F^{\mathbf{35}}$ by inverting (D.17):

$$F^{\mathbf{35}}_{mn}=f^{\mathbf{35}}_{mn}-\frac{2L^if^{\mathbf{35}}_{i(m}L_{n)}}{1+L^2}$$
$$+\frac{L^iL^jf^{\mathbf{35}}_{ij}}{(1+L^2)(1+\frac{3}{4}L^2)}\left\{\frac{9}{14}L_mL_n+\frac{1}{4}g_{mn}(1+\frac{3}{7}L^2)\right\}\,. \tag{D.19}$$

Note that $F^{\mathbf{35}}_{mn}$ in (3.17) is traceless by virtue of

$$\left\{g^{mn}-2\frac{L^mL^n}{(1+L^2)}\right\}\nabla_mL_n-4m(1-L^2)+12L^i\partial_i\Delta=0\,, \tag{D.20}$$

which is obtained by tracing equation (D.3). A straightforward manipulation then leads to the second line of equation (3.16).

The **7** and **21** parts of (D.3) are treated similarly. Taking the identities

$$0=(P^{\mathbf{7}})^{pq}_{rs}\left\{\Phi_p{}^{ijk}\omega^2_{qij}L_k+4\omega^2_{pqi}L^i\right\}$$
$$0=(P^{\mathbf{21}})^{pq}_{rs}\left\{\Phi_p{}^{ijk}\omega^2_{qij}L_k\right\} \tag{D.21}$$

into account, we arrive at

$$(\nabla_{[r}L_{s]})^{\mathbf{7}}=(P^{\mathbf{7}})^{mn}_{rs}\left\{L^i\nabla_iL_mL_n-\frac{1}{2}L_m\partial_nL^2+3(1+L^2)L_m\partial_n\Delta\right\} \tag{D.22}$$

$$(\nabla_{[r}L_{s]})^{\mathbf{21}}=(P^{\mathbf{21}})^{mn}_{rs}\left\{L^i\nabla_iL_mL_n-\frac{1}{2}L_m\partial_nL^2+3(1+L^2)L_m\partial_n\Delta\right\}\,. \tag{D.23}$$

It is then straightforward to show that the equations above are equivalent to the first line of (3.16). Taking all the above into account, the expressions for the remaining flux components f, $F^{\mathbf{1}}$, $F^{\mathbf{7}}$ are obtained by straightforward manipulations of equations (D.4), (D.5), (D.6), respectively.

E. Spinor vs four-form

In this section we prove the equivalence of the two equations in (3.15). One needs to show that the existence of the self-dual four-form Φ is equivalent to the existence of the chiral spinor η (see e.g. [23]). The two equations in (3.15) are then essentially equivalent to the statement that Φ, η are acted upon by the Levi-Civita, the (associated) spin connection respectively, and that they are both Spin(7) singlets. More explicitly,

($\Longrightarrow$): Let us expand

$$\nabla_m \eta = A_m \eta + B_{m,pq}\gamma^{pq}\eta\,, \tag{E.1}$$

where without lost of generality, as can be seen from the first line of equation (B.2), $B_{m,pq}$ can be taken to satisfy

$$B_{m,pq} = (P^{\mathbf{7}})_{pq}^{rs} B_{m,rs}\ . \tag{E.2}$$

From (E.2) we can see immediately that

$$4B_{[m,pq]} = B_{i,j[m}\Phi^{ij}{}_{pq]} - \Phi_{mpq}{}^{j}B^{i}{}_{,ij} \tag{E.3}$$

$$B_{i,jk}\Phi^{ijk}{}_{m} = -6B^{i}{}_{,im}\,, \tag{E.4}$$

from which it follows that

$$B_{m,pq}\gamma^{pq}\eta = 6B_{[m,pq]}\gamma^{pq}\eta + 4B^{p}{}_{,pq}\gamma^{q}{}_{m}\eta\ . \tag{E.5}$$

Moreover, the fact that η has unit norm implies $A_m = 0$. On the other hand, it follows from the first line in (3.15) and (E.1) that

$$\omega^1_m = -\frac{8}{7}B^{i}{}_{,im} \tag{E.6}$$

$$\omega^2_{mpq} = 6(B_{[m,pq]} + \frac{1}{7}\Phi_{mpq}{}^{j}B^{i}{}_{,ij})\ . \tag{E.7}$$

Taking (E.4) into account, we can see that the right-hand side of (E.7) transforms in the **48** of Spin(7), as of course it should. Collecting all the above, the second line of equation (3.15) follows.

($\Longleftarrow$): It can be shown that the four-form Φ satisfies the following useful identity [37]

$$\frac{1}{24}\varepsilon_{mnpq}{}^{ijkl} = \frac{1}{168}\Phi_{mnpq}\Phi^{ijkl} + \frac{3}{28}\Phi_{[mn}{}^{[ij}\Phi^{kl]}{}_{pq]} + \frac{2}{21}\Phi^{[i}{}_{[mnp}\Phi_{q]}{}^{jkl]}\ . \tag{E.8}$$

Moreover, any S_{mnp} in the **48** of Spin(7) satisfies

$$\begin{aligned} S_{mnp} &= \frac{3}{2}\Phi^{ij}{}_{[mn}S_{p]ij} \\ \Phi_m{}^{ijk}S_{ijk} &= 0\ . \end{aligned} \tag{E.9}$$

Using (E.8), (E.9), we can see that

$$\varepsilon_{mnpqr}{}^{ijk}\omega^2_{ijk} = 60\Phi_{[mnp}{}^{i}\omega^2_{qr]i}\ . \tag{E.10}$$

Contracting the second line of (3.15) with $\eta\gamma_{pqrs}$ and using (E.10), (3.14), the first line of equation (3.15) follows.

F. Generalized Spin(7) structures

In this appendix we include the details of the derivation of equation (4.8). Multiplying (3.7), (3.8) on the left by $\xi^+\gamma_m$, $\xi^-\gamma_m$ respectively and subtracting, we obtain

$$F_{mijk}\widehat{\Psi}^{ijk} = 36\Psi^-\partial_m\Delta - 12\Psi^+ f_m + 72m\widehat{\Psi}_m \ . \tag{F.1}$$

Multiplying (3.5), (3.6) on the left by ξ^+, ξ^- respectively and adding/subtracting, taking (F.1) into account, we obtain

$$\begin{aligned} 0 &= \partial_m\Psi^+ \\ 0 &= e^{-3\Delta}\partial_m(e^{3\Delta}\Psi^-) - \Psi^+ f_m + 4m\widehat{\Psi}_m \ . \end{aligned} \tag{F.2}$$

Multiplying (3.7), (3.8) on the left by $\xi^+\gamma_{mpqrs}$, $\xi^-\gamma_{mpqrs}$ respectively and adding/subtracting, we obtain

$$\begin{aligned} \widehat{\Psi}_{[mpq}{}^{ij}F_{rs]ij} &= -6\Psi^+{}_{[mpqr}\partial_{s]}\Delta + 2\Psi^-{}_{[mpqr}f_{s]} + \widehat{\Psi}_{[m}F_{pqrs]} \\ \widehat{\Psi}_{[mp}{}^{i}F_{qrs]i} &= -3\Psi^-{}_{[mpqr}\partial_{s]}\Delta + \Psi^+{}_{[mpqr}f_{s]} + \frac{1}{12}\widehat{\Psi}_{[mpqr}{}^{ijk}F_{s]ijk} - \frac{6}{5}m\widehat{\Psi}_{mpqrs} \ . \end{aligned} \tag{F.3}$$

Multiplying (3.5), (3.6) on the left by $\xi^+\gamma_{pqrs}$, $\xi^-\gamma_{pqrs}$ respectively and adding/subtracting, taking (F.3) into account as well as the identity

$$\widehat{\Psi}_{[mpqr}{}^{ijk}F_{s]ijk} = -6\widehat{\Psi}_{[m}(\star F)_{pqrs]} \ , \tag{F.4}$$

we obtain

$$\begin{aligned} 0 &= e^{-6\Delta}\partial_{[m}(e^{6\Delta}\Psi^+{}_{pqrs]}) + F_{[mpqr}\widehat{\Psi}_{s]} \\ 0 &= e^{-9\Delta}\partial_{[m}(e^{9\Delta}\Psi^-{}_{pqrs]}) + (\star F)_{[mpqr}\widehat{\Psi}_{s]} - \Psi^+{}_{[mpqr}f_{s]} + \frac{8}{5}m\widehat{\Psi}_{mpqrs} \ . \end{aligned} \tag{F.5}$$

Rescaling the metric $g_{mn} \to g'_{mn} := e^{-3\Delta}g_{mn}$ has the effect that the gamma matrices also get rescaled as: $\gamma^m \to e^{3\Delta/2}\gamma^m$. Passing to the bispinor notation in the rescaled metric g'_{mn}, equations (F.2), (F.5) can be written succinctly as

$$\begin{aligned} 0 &= d\Psi^+ + F\wedge\widehat{\Psi}^- \\ 0 &= e^{-3\Delta}d(e^{3\Delta}\Psi^-) + \star F\wedge\widehat{\Psi}^- - f\wedge\Psi^+ + 4m\widehat{\Psi}^- \ , \end{aligned} \tag{F.6}$$

where

$$\widehat{\Psi}^- := \frac{1}{2}(\widehat{\Psi} - \star\widehat{\Psi}) = \frac{1}{8}P_+\left\{\widehat{\Psi}_m\gamma^m + \frac{1}{5!}\widehat{\Psi}_{m_1\dots m_5}\gamma^{m_1\dots m_5}\right\} \ . \tag{F.7}$$

We also define

$$\widehat{\Psi}^+ := \frac{1}{2}(\widehat{\Psi} + \star\widehat{\Psi}) = \frac{1}{8}P_+\left\{\frac{1}{3!}\widehat{\Psi}_{m_1\dots m_3}\gamma^{m_1\dots m_3} + \frac{1}{7!}\widehat{\Psi}_{m_1\dots m_7}\gamma^{m_1\dots m_7}\right\} \ . \tag{F.8}$$

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Multiplying (3.7), (3.8) on the left by $\xi^-\gamma_{mp}$, $\xi^+\gamma_{mp}$ respectively and adding, we obtain

$$\frac{1}{12}F_{[m}{}^{ijk}\Psi^+{}_{p]ijk} + \frac{1}{2}f_i\widehat{\Psi}^i{}_{mp} = -3\widehat{\Psi}_{[m}\partial_{p]}\Delta \ . \tag{F.9}$$

Multiplying (3.5), (3.6) on the left by $\xi^-\gamma_p$, $\xi^+\gamma_p$ respectively and adding, taking (F.9) into account, we obtain

$$0 = \partial_{[m}(e^{3\Delta}\widehat{\Psi}_{p]}) \ . \tag{F.10}$$

Multiplying (3.7), (3.8) on the left by $\xi^-\gamma_{mpqr}$, $\xi^+\gamma_{mpqr}$ respectively and subtracting, we obtain

$$\frac{3}{4}\Psi^-{}_{[mp}{}^{ij}F_{qr]ij} + \frac{1}{2}\widehat{\Psi}_{mpqr}{}^i f_i = -6\widehat{\Psi}_{[mpq}\partial_{r]}\Delta + \frac{1}{4}(\star F)_{mpqr}\Psi^+ + \frac{1}{4}F_{mpqr}\Psi^- + 3m\Psi^+{}_{mpqr} \ . \tag{F.11}$$

Multiplying (3.5), (3.6) on the left by $\xi^-\gamma_{pqr}$, $\xi^+\gamma_{pqr}$ respectively and adding, taking (F.11) into account as well as the identity

$$\frac{1}{24}\Phi^{\pm}_{mpqr}{}^{n_1\dots n_4}F_{n_1\dots n_4} = \pm\Phi^{\pm}(\star F)_{mpqr} \ , \tag{F.12}$$

we obtain

$$0 = e^{-6\Delta}\partial_{[m}(e^{6\Delta}\widehat{\Psi}_{pqr]}) + \frac{1}{4}(\star F)_{mpqr}\Psi^+ - \frac{1}{4}F_{mpqr}\Psi^- + m\Psi^+{}_{mpqr} \ . \tag{F.13}$$

Multiplying (3.7), (3.8) on the left by $\xi^-\gamma_{mpqrst}$, $\xi^+\gamma_{mpqrst}$ respectively and subtracting, we obtain

$$5\Psi^+{}_{[mpq}{}^iF_{rst]i} + \frac{1}{2}\widehat{\Psi}_{mpqrst}{}^i f_i = -9\widehat{\Psi}_{[mpqrs}\partial_{t]}\Delta + \frac{1}{4}\Psi^+{}_{[mpqrs}{}^{ijk}F_{t]ijk} \ . \tag{F.14}$$

Multiplying (3.5), (3.6) on the left by $\xi^-\gamma_{pqrst}$, $\xi^+\gamma_{pqrst}$ respectively and adding, taking (F.14) into account as well as the identity

$$0 = \Psi^{\pm}_{[mpqrs}{}^{ijk}F_{t]ijk} \ , \tag{F.15}$$

we obtain

$$0 = \partial_{[m}(e^{9\Delta}\widehat{\Psi}_{pqrst]}) \ . \tag{F.16}$$

Multiplying (3.7), (3.8) on the left by $\xi^-\gamma_{mpqrstuv}$, $\xi^+\gamma_{mpqrstuv}$ respectively and subtracting, we obtain

$$12\widehat{\Psi}_{[mpqrstu}\partial_{v]}\Delta = \frac{35}{2}\Psi^-{}_{[mpqr}F_{stuv]} + 3m\Psi^+{}_{mpqrstuv} \ . \tag{F.17}$$

Multiplying (3.5), (3.6) on the left by $\xi^-\gamma_{pqrstuv}$, $\xi^+\gamma_{pqrstuv}$ respectively and subtracting, taking (F.17) into account, we obtain

$$0 = \partial_{[m}(e^{12\Delta}\widehat{\Psi}_{pqrstuv]}) + m\Psi^+{}_{mpqrstuv} \ . \tag{F.18}$$

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Passing to the bispinor notation for the rescaled metric, taking into account the identity

$$(\star F)_{[mpqr}\Psi^{+}{}_{stuv]} - F_{[mpqr}\Psi^{-}{}_{stuv]} = 0\,, \tag{F.19}$$

equations (F.10), (F.13), (F.16), (F.18) can be written succinctly as

$$\begin{aligned} 0 &= e^{-3\Delta/2}d(e^{3\Delta/2}\widehat{\Psi}^{-}) \\ 0 &= e^{-3\Delta/2}d(e^{3\Delta/2}\widehat{\Psi}^{+}) + 2(\star F\wedge\Psi^{+} - F\wedge\Psi^{-}) + 8m\Psi^{+}\,. \end{aligned} \tag{F.20}$$

It is easy to see that the integrability of (F.6), (F.20) follows from the equations-of-motion and Bianchi identities. The 'asymmetry' between $\Psi^{\pm}$ in equation (F.6) disappears in the massless limit for constant warp factor and in the absence of fluxes along the noncompact spacetime directions; i.e. for Δ =constant, m, $f=0$. In this case we have:

$$\begin{aligned} 0 &= d\Psi^{+} + F\wedge\widehat{\Psi}^{-} \\ 0 &= d\Psi^{-} + \star F\wedge\widehat{\Psi}^{-} \\ 0 &= d\widehat{\Psi}^{-} \\ 0 &= d\widehat{\Psi}^{+} + 2(\star F\wedge\Psi^{+} - F\wedge\Psi^{-}) \end{aligned} \tag{F.21}$$

The integrability of the above equations follows from the equations-of-motion and Bianchi identities, which now read dF, $d\star F=0$. Note, however, that setting $m=0$ excludes any solutions with nonzero fluxes if X is smooth and compact, as was already noted in section 3.

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